Mathematical Study for the Outflow of Aqueous Humor and Function in the Eye

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Abstracts: A study of aqueous humor outflow passing through the trabecular meshwork in the canal of Schlemm's is presented in this paper. Canal of Schlemm's is connected by the trabecular meshwork and that passage is considered as a permeable compliant channel. The internal wall of the canal is porous as well as aqueous and humor percolates along it. The aqueous humor is filtering through the canal segment before attaining to the collector channel. This present model suggested flow in a canal section in the middle of two different collector channels. First one is Elliptic passage collector channel and the second is annular passage collector channel. Aqueous humor fluid pressure as well as volume flux and the effects of filtration constant, intraocular pressure are discussed in this present paper. Due to the increased intraocular pressure the more aqueous humour has to percolate into the canal via the permeable innermost layer and this extra percolation affects the volume flux of aqueous humour in the canal of Schlemm. It is observed in this work that aqueous flux rises with the increased percolation of aqueous humour inside the Schlemm's canal.

Keywords: Canal of Schlemm's, Trabecular meshwork, Aqueous humor flow, Collector channel, Aqueous fluid pressure.

Introduction: Primary Open Angle Glaucoma (POAG) give rise to blindness and continuously affecting 70–75 million patients globally. In order to understand, Primary Open Angle Glaucoma (POAG), it is important to understand the anatomy of eye, formation and drainage of aqueous humor (AH) in the eye [Fig. 1]. The eyes are organs of the visual system and aqueous humor is an optically clear, slightly alkaline ocular thin fluid like water, it is similar to the plasma of the blood and located between the anterior chambers and posterior chamber of the eye. Aqueous humor flows through the ciliary fibers, towards the posterior chamber in the middle of the lens and the iris. From this chamber the aqueous humor fluid flows out of the pupil and get into the anterior chamber. The ciliary body build around 2.5 μL of aqueous humor each minute and the ciliary body continuously produces aqueous humor in the eye by the ciliary processes. It is transparent substance continuously produced by the ciliary epithelium approximately 2.3 µlitre/min. It flows behind the iris in the posterior chamber and drains from the eye through the drainage angle. Aqueous humor start flowing from that angle and passes by way of biological filter which is known as trabecular meshwork (TM) toward the Schlemm's of Canal (SC), which is the main drainage passage from the eye and lastly reaches to the "collector channels". Regular production and drainage occurrence of this aqueous humor fluid in the eye is an essential process to ensure the stability of the IOP which is very necessary to support the visual activity of eye and to provide the nutrition to the tissues to the eye. The average intraocular pressure of a healthy human is about 12-22 mmHg. In health eye, the rate of secretion of aqueous humor balances the rate of drainage of aqueous humor. When this drainage process somehow shut off the aqueous humor cannot go out quickly hence the intraocular pressure (IOP) rises in eye. The excessive production of aqueous fluid in the anterior chamber of the eye raises the intraocular pressure (IOP). If this raises intraocular pressure retained for very long period then it leads to destruct the optic nerve and may cause the Primary

Open Angle Glaucoma, which is the main reason of loss of sight. The patients with Glaucoma, the drainage of aqueous humor from the trabecular meshwork is partially or entirely choked. Aqueous humor fluid increases in the chambers and due to this the pressure increases inside eye. This pressure pushes lens system in the posterior body

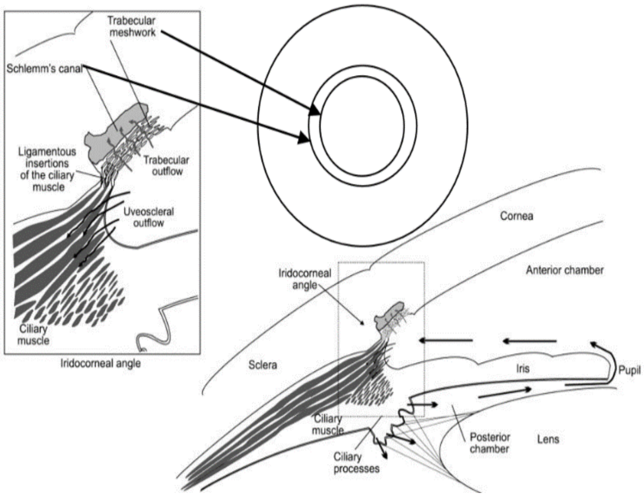

Fig. (1). Anatomy of eye

and the lens pushes the vitreous body which compresses the eye and as a result the blood vessels of the eye and nerves fibers are disturbed. These disturbed and damaged blood vessels and nerves fibers are result in Glaucoma. Primarily, the glaucoma is of two types. (1) Open angle glaucoma and (2) Closed angle glaucoma. The open angle glaucoma retains lifelong in the eye. It is initiated by partially blocking of the drainage of the canal. The angle between the iris and the cornea is wide open that means the entrance at the drainage is unblocked, but the outflow of aqueous humor fluid is slightly sluggish. The pressure increases progressively in the eye for too long period. Symptoms start appearing gradually from peripheral vision loss and can go on till the loss of central vision. Growth of glaucoma can be pause with medical treatments, but part of the image that has already been lost cannot get back. This is why it is crucial to detect the signs of glaucoma early with the regular eye exams. The closed angle glaucoma is inspired by a sudden complete blockage of AH drainage. The pressure in the eye rises rapidly and leads to vision loss rapidly. This happens when pupil dilated and stuck to the back of the iris. This prevents the AH from flow into the Anterior Chamber. Accumulation of the aqueous humor into the posterior chamber presses on the iris results in the block the angle fully. This Closed Angle Glaucoma is a medical emergency and required instant attention. In both cases, intraocular pressure (IOP) raises. In the case of Open-Angle Glaucoma, the most effective technique to lower down the IOP is by attacking the inflow system with drugs that decreases the rate of production and secretion of aqueous humor. Aqueous humor flows through the outflow network at a shallow rate of flow (2.0 µl/min), still this flow built an unexpectedly significant pressure drop throughout a short flow distance (less than 1 mm). In a healthy human eye, the pressure drop is around 6 mmHg and it can be as much as 40 mmHg in the condition of glaucoma in the human eye. Several studies have done to determine the principal site of flow resistance in the aqueous outflow network. Therefore, it is required to enrich the knowledge and developed the research models for the factors contributing to the collapse of the canal and associated mechanism. The experimental investigations are also needed in this field. Consequently, for the better understanding of the outflow of aqueous humor and network as well as changes in the flow, which can lead to Primary Open Angle Glaucoma (POAG). We have done this present research work, in which we have studied the involvement of Schlemm's canal in buildup volume flux and pressure drop in the normal eye with different shape of passage of tube. This model depicts the aqueous humor out flow from the AC through the TM and into the SC and the predictions of changes in volume flux and intraocular pressure (IOP).

Mathematical Formulation of the problem: Nearly all of the aqueous humor fluid running through canal of Schlemm has to flow some extent along the canal to get into the collector channel. There are two kinds of channel represented in this model, circular channel and elliptical channel between the two collectors of a canal [Fig. 2, 4]. Almost half of the total amount of the humor percolates through the canal and reaches the collector to the right of the midpoint between two collectors and another half to the left side. Hence the flow at the midpoint, described by z = 0, is zero [Fig. 3]. Thus, aqueous humor flow in the section is regarded as fluid flow through a narrow shape of elliptic and annular passage. The following assumptions have taken in the formulation of the problem:

a) The flow is taken in to consideration in half of the segment in z direction due to symmetry.
b) The damage in the inner layer of canal of Schlemm is proportional to the pressure drop.
c) The aqueous fluid flow is Newtonian, laminar, steady, incompressible and viscous.
d) The size of the collector channels is same.
e) All collector channels are draining the same amount of the aqueous humor fluid.
f) The innermost endothelium layer of the canal is permeable, collapsible and porous.
g) All the collector channels are on the equal distance.

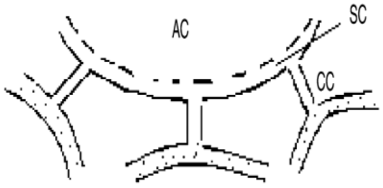

Fig. (2).Canal of Schlemm's with collector channel

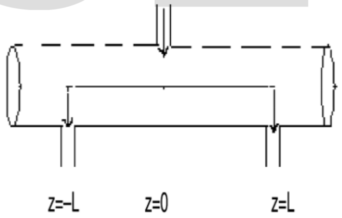

Fig. (3). Segment of canal as an elliptic channel with upper wall permeable

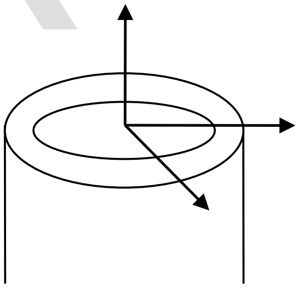

Fig. (4). The canal segment with an elliptic cross-section

The governing equations of fluid mechanics (continuity and motion):

The governing Navier-Stokes equation which is a partial differential equation stating the local balance of the momentum in the fluid around any time any point in the space is give n by:

$$\rho \frac{\mathrm{Dv}}{\mathrm{Dt}} = -\nabla p + \mu \nabla^2 v + f \tag{1}$$

Where, $\frac{Dv}{Dt} = Instantenious$
$-\nabla p = DynamicPressure$
$\mu \nabla^2 v = Shearviscosityterm$
$f = Externalforce$

$\rho = Density$

By introducing the assumptions in the equation (1), the leading Navier–Stokes equations represented as:

$$\frac{-\partial p}{\partial x} = 0 \tag{2}$$

$$\frac{-\partial p}{\partial y} = 0 \tag{3}$$

$$\frac{-\partial p}{\partial z} + \mu \Delta^2 v_z = 0 \tag{4}$$

The equation of continuity is reducing as:

$$\frac{-\partial v_z}{\partial z} = 0 \tag{5}$$

From material balance equation:

$$w(z) = q(z + dz) - q(z) \tag{6}$$

Where, $w(z)$ denotes the filtration flux of the aqueous humour and $q(z)$ denotes the aqueous humour volume flux in the canal.

By expansion of $q(z + dz)$ in a Taylor series, we have:

$$\frac{dq(z)}{dz} = w(z) \tag{7}$$

where, $q(z) = \iint v_z \, \mathrm{dxdy},$
$w(z) = G[P_1 - p(z)]$

The boundary conditions are given below:

$v_z(x, y) = 0$

around the inner perimeter,

$$p(z = L) = P_0 \qquad \frac{\mathrm{dp}}{\mathrm{dz}}_{z=0} = 0 \tag{8}$$

where P_0 is the pressure at the entrance.

The solution to the problem:

Case I: Elliptic passage: The equation no. (1) and (2) represent that the pressure, the governing differential equation is reduced to the form:

$$\frac{1}{\mu}\frac{\partial p}{\partial z} = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} \tag{9}$$

For the elliptic boundary v_z is

$$v_z = k\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{10}$$

Using Equation no. (12) in Equation. (11),

$$k = \frac{1}{2\mu}\left(\frac{a^2b^2}{a^2+b^2}\right)\frac{\mathrm{dp}}{\mathrm{dz}} \tag{11}$$

By using the value of k in equation (10),

$$v_z = \frac{1}{2\mu}\left(\frac{a^2b^2}{a^2+b^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\frac{\mathrm{dp}}{\mathrm{dz}} \tag{12}$$

And

$$q(z) = \iint v_z \, \mathrm{dxdy} = \frac{\pi \mathrm{abk}}{2} \tag{13}$$

Aqueous out flux is:

$$q(z) = \frac{1}{4\mu}\left(\frac{\pi a^3 b^3}{a^2+b^2}\right)\frac{\mathrm{dp}}{\mathrm{dz}} \tag{14}$$

The elliptical boundary, from Eq. (6)

$$\frac{\mathrm{dq}}{\mathrm{dz}} = \frac{\pi}{2}(a + b)w(z) \tag{15}$$

From boundary conditions in equation no. (8) and (13), differential equation is as below:

$$\frac{d^2p}{\mathrm{dz}^2} - m^2 p(z) = m^2 P_1. \tag{16}$$

where, $m = \left(\frac{2\mu(a+b)(a^2+b^2)}{a^3b^3}G\right)^{1/2}$

The solution of Eq. (16) with the boundary conditions is given as:

$$p = P_1 - \frac{P_1 - p_0}{\cosh(\mathrm{ml})}\cosh(\mathrm{mz}) \tag{17}$$

And also

$$q(z) = \frac{\mathrm{ml}}{4\mu}\left(\frac{\pi a^3 b^3}{a^2+b^2}\right)\left(\frac{P_1-p_0}{\cosh(\mathrm{ml})}\sinh(\mathrm{mz})\right) \tag{18}$$

$$w(z) = \frac{m^2}{2\mu(a+b)}\left(\frac{a^3 b^3}{a^2+b^2}\right)\left(\frac{P_1-p_0}{\cosh(\mathrm{ml})}\cosh(\mathrm{mz})\right) \tag{19}$$

Case II: Annular passage: In this case, it is assuming a = b = r and now expression for volume flux and pressure are:

$$\frac{d^2p}{\mathrm{dz}^2} - m^2 p(z) = m^2 P_1 \tag{20}$$

where $m = (\frac{8\mu}{r^3} G)^{1/2}$

and r is the canal's radius.

To find the solution of Eq. (17), we have,

$$p = P_1 - \frac{P_1-p_0}{\cosh(\mathrm{ml})}\cosh(\mathrm{mz}) \tag{21}$$

$$q(z) = \frac{\pi r^4 m}{8\mu}\left(\frac{P_1-p_0}{\cosh(\mathrm{ml})}\sinh(\mathrm{mz})\right) \tag{22}$$

and

$$w(z) = \frac{m^2 r^2}{4\mu}\left(\frac{P_1-p_0}{\cosh(\mathrm{ml})}\cosh(\mathrm{mz})\right) \tag{23}$$

Results and discussion: The model shown before contributes to the fact that the increased aqueous flux and pressure contribute in damaging the optic nerve that results in Glaucoma. Therefore, the incorporation of elliptic passage cross-sectional tube and annular passage cross-sectional tube describe the simplest representation of aqueous humour flow in the eye from canal of Schlemm's segment to collator channel through trabecular meshwork. The effects of significant parameters like intraocular pressure of normal eye, filtration constant on pressure profile and volume flux profile in the elliptic and circular passage canal have investigated and discussed. The analytical results obtained in this study consist of the expression for volume flux and pressure in two different cases for elliptical and circular cross-sectional tube in equation no. (18), (19), (22), (23) and display through graphs from Fig. (5) to Fig. (12). The two different cases are:

(i) The porous inner layer of the canal passage is collapsible and change into shape of an elliptic passage tube.

(ii) The porous inner layer of canal passage is rigid and change into the shape of an annular passage tube.

The different values of parameters used in this present paper are shown in Table 1.

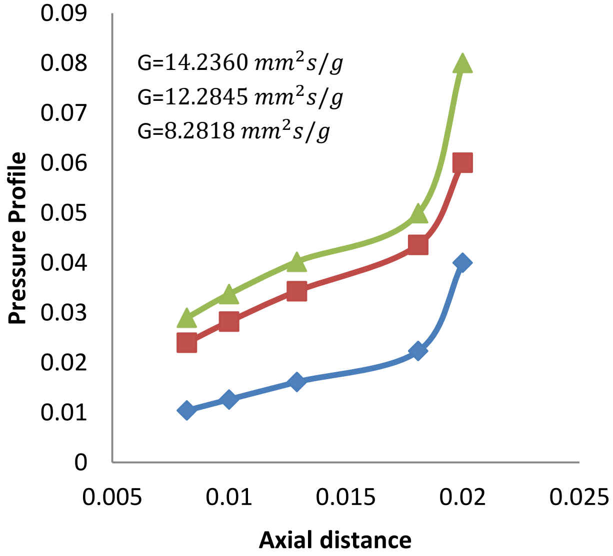

Fig. (5): Pressure profile with axial distance for different values of filtration constant in elliptical channel

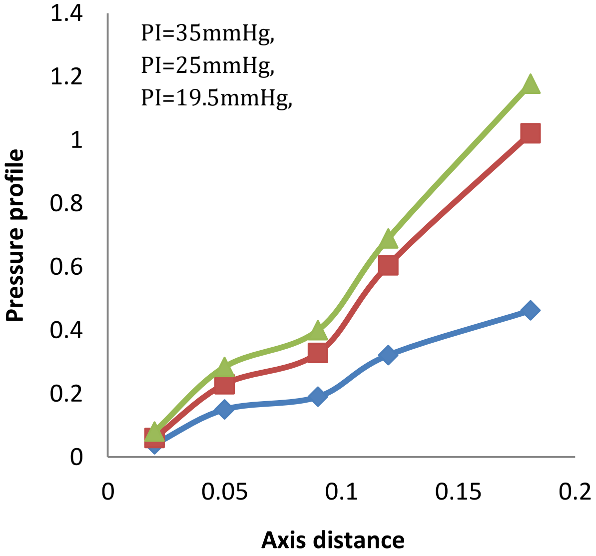

Fig. (6): Pressure profile with axial distance for different values of intraocular pressure in elliptical channel

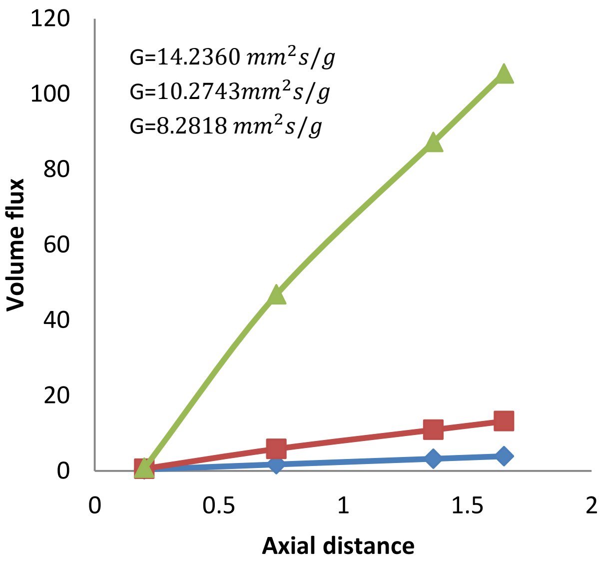

Fig. (7): Volume flux with axial distance for different values of filtration constant in elliptical channel

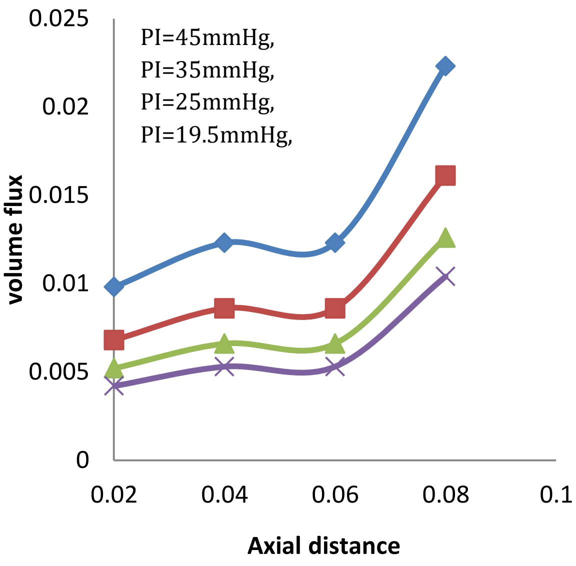

Fig. (8): Volume flux profile with axial distance for different values of intraocular pressure in elliptical channel

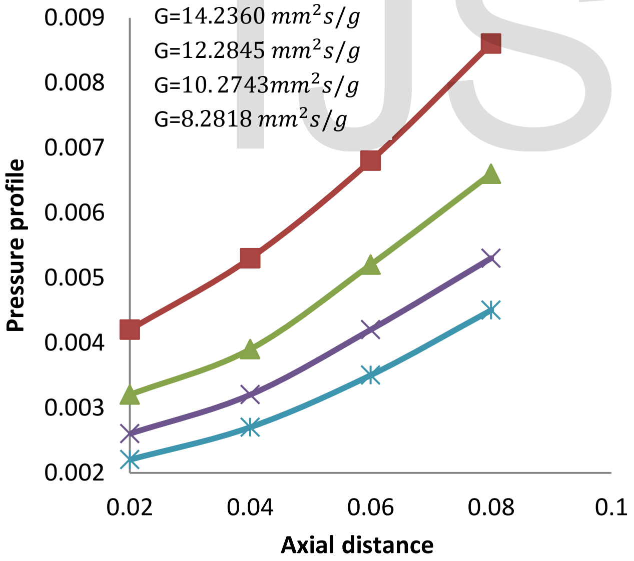

Fig. (9): Pressure profile with axial distance for different values of filtration constant in circular channel

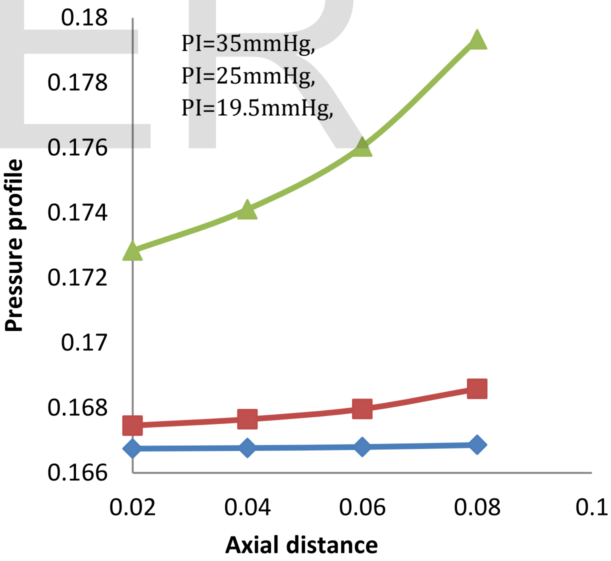

Fig. (10): Pressure profile with axial distance for different values of intraocular pressure in circular channel

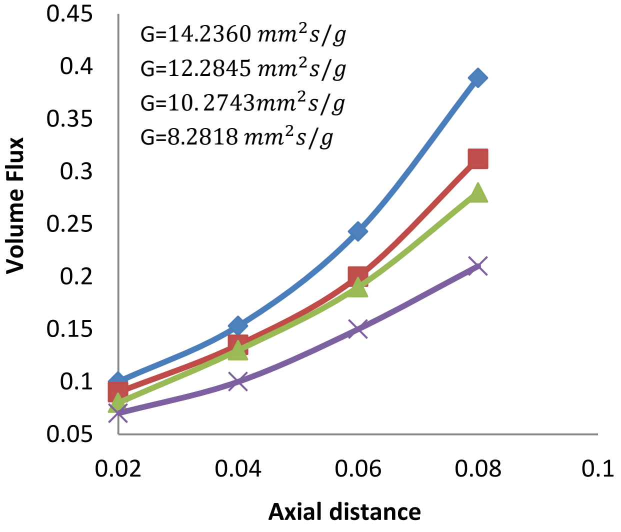

Fig. (11): Volume flux with axial distance for different values of filtration constant in circular channel

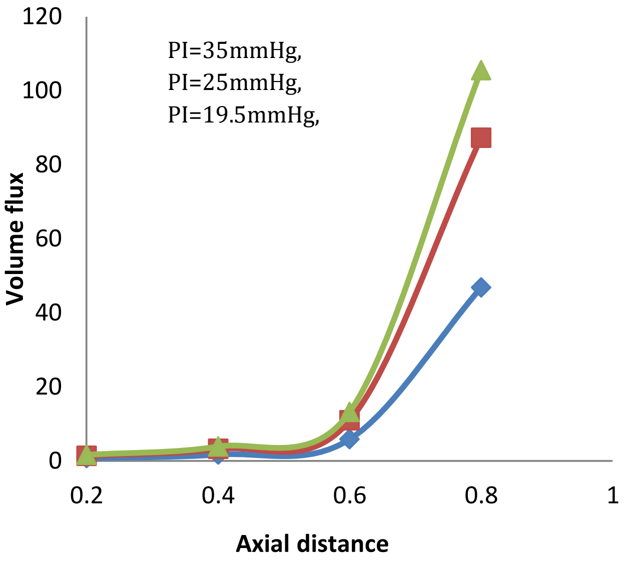

Fig. (12): Volume flux profile with axial distance for different values of intraocular pressure in circular channel

Table 1: Value of parameter and description for aqueous humour flow

Parameter	Standard Value
Aqueous humour pressure in collector channel (P_0)	$12mmHg$
Filtration constant (G)	8.28182 mm^2s/g
Distance between two collector channels ($2l$)	$1200\mu m$
Dynamic viscosity of aqueous humour (μ)	$0.75cp$
Value of parameter for major axis of the elliptic passage (a)	$0.15mm$
Value of parameter for minor axis of the elliptic passage (b)	$0.005mm$
Value of parameter for radius of the annular passage (r)	$0.13mm$

Fig. (5) to Fig. (8) consist the variation of pressure profile, volume flux with axial distance for different parameters of filtration constant and intraocular pressure in the elliptical channel. And the variation of pressure profile, volume flux with axial distance for different parameters of filtration constant and intraocular pressure in circular channel have shown by figures from Fig. (9) to Fig. (12). It is evident from the Fig. (5), Fig. (6), Fig (9) and Fig. (10) that the pressure increases and volume flux increases as distance increases. It should also be noted here that the pressure and volume flux increase as filtration constant and intraocular pressure increases. The results are therefore consistent with the observation of [10]. Fig. (7), Fig. (8), Fig. (11) and Fig. (12) describe that the volume flux and pressure increase as the filtration constant and intraocular pressure increases. It is also noticed here that the pressure and volume flux increase as distance increases and these results are compared with the [7]. The aqueous humour volume flux rises in Schlemm's canal with an increase of intraocular pressure. The filtration constant rises as well as the aqueous percolation along the inner layer of the wall rises. And due to this extended percolation, the aqueous flux raises in the canal and it is demonstrated by the curves in Fig. (7) and Fig. (8). The volume flux, as is detected from graph in Fig. (11) and Fig. (12) that when (IOP) will increase or arise, the filtration constant (G) forces enough and more aqueous humour to percolate into the annular canal through the porous inner wall and percolation through the inner wall will be increased. The pressure and the volume flux are more in the elliptical channel than the circular channel. It is clear from the results that a slight change in the cross-sectional of tube bring about a noticeable change in the aqueous flux and pressure.

Conclusion: Due to the increased intraocular pressure the optic nerve damaged and results in Glaucoma. In this case the more aqueous humour has to percolate through the canal passage into the porous inner layer and this developed percolation affects the aqueous volume flux of canal. The aqueous flux rises with increased percolation of the aqueous humour in the canal. The pressure profile and the volume flux are more for the elliptic passage channel than the annular passage channel.

References:

1. A. Bill, O. Maepea, T. Hamanaka, Aspect of aqueous humour drainage through Schlemm’s canal, Klin. Monatsbi Augenheilbed 195, 5, pp 277–280, 1989.
2. A.P. Nesterov, Role of blockage of Schlemm_s canal in the pathogenesis of primary open-angle glaucoma, Am. J. Ophthalmol. 70 pp 691–696, 1970.
3. A.P. Nesterov, Y.E. Batmanov, Trabecular meshwork of Schlemm canal in the early stage of primary open glaucoma, Am. J. Ophthalmol. 78 pp. 639–647, 1974.
4. C.R. Ethier, Biomechanics, Aqueous humor drainage and glaucoma, A lecturer delivered in Application of Mathematics in Medicine. Summer School and Workshop Organized at The field Institute, Toronto, Ontario, Canada, pp. 21–30, 2003.
5. C.R. Ethier, The inner wall of Schlemm’s Canal, Exp. Eye Res. Rev. 74 (2) (2002) 161–172.
6. C.R. Greaney, M.J. Dewyne, A.D. Fitt, Fluid flow in the Anterior chamber of a human eye, IMA J. Math. Appl. Med. Biol. 19, pp 31–60, 2003.
7. E.M. Van Buskirk, An atomic correlation of changing aqueous outflow of the facility in excised human eyes, Invest. Ophthalmol. Visual Sci. 225, pp. 625–632, 1982.
8. I. Grierson, Effect of increased intraocular pressure on the outflow approaches, Exp. Eye Res. 18, pp. 413–419, 1979.
9. M. Johnson, R.D. Kamm, The role of Schlemm_s Canal in the aqueous outflow from the human eye, Invest. Ophthalmol. Vis. Sci. 24, 320, 1983.
10. M.A. Johnson, W.M. Grant, Pressure dependent changes in structures of the aqueous outflow system of human and monkey eyes, Am. J. Ophthalmol. 75, pp. 365–383, 1973.
11. R.A. Moses, The effect of intraocular pressure on the resistance of outflow, Survey Ophthalmol. 22 pp. 88–100, 1977.
12. R.A. Moses, W.J. Grodzki Jr., E.L. Etheridge, C.D. Wilson, Schlemm_s Canal: The effect of intraocular pressure, Invest. Ophthalmol. Vis. Sci. 20, 1 pp. 61–68, 1981.
13. R.F. Brubaker, The flow of aqueous humor in a human eye, Tr. Am. Ophthalmol. Soc. 80 pp.391–474, 1982.